Nonequilibrium phase transition in the case of correlated noises

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We consider the nonequilibrium phase transition in the case of correlated noises. A spatially extended model driven by the correlated noises is studied. By the Weiss mean-field approximation, we find that the correlation between the additive and multiplicative noises has an effect on the phase transition. The phase transition possesses some similar characteristics to the one of Van den Broeck, Parrondo, and Toral [Phys. Rev. Lett. **73**, 3395 (1994)]. In addition, we investigate the effect of the additive noise on the phase transition.

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Recently, the noise-induced phase transition has been intensively investigated for a large number of systems [1–5]. In Ref. [2], GarcíaOjàlvo, Hernández-Machado, and Sancho have presented a spatially extended model, i.e., the Swift-Hohenberg equation, and studied the influence of multiplicative noise on the evolution of microscopic variables. Subsequently, Van den Broeck and collaborators proposed two general spatially extended models that describe the effect of multiplicative noise [4,5], and reported that the models can undergo a nonequilibrium phase transition leading to a symmetry-breaking state. In these systems, the noise plays an important role. Noise-induced transitions can occur only if a certain amount of randomness is present in the environment. Remarkably they amount to a symbiotic relationship of order and randomness. The existence of noise-induced transitions clearly forces us to appraise the role of noise. As we know, noise always has internal and external origins. So for a system driven by noises, we should simultaneously consider the additive noise (internal noise) and the multiplicative noise (external noise). In the case of independent noises, the stationary properties of the system are dominated by the multiplicative noise and the role of the additive one becomes rather small. The authors of Refs. [6–8] have investigated the effect of interference of additive noise and multiplicative noise when these two kinds of noises are correlated. They found that the transition between unimodal and bimodal stationary distribution (i.e., the transition between stationary states of the system) is strongly influenced by the correlation between both noises. So one can ask how the effect of the correlation on nonequilibrium phase transitions will be. In this paper, we shall study this problem.

We consider the lattice model with scalar variables, in which the scalar variables x_i are defined on lattice points i ($i=1,2,\ldots,L^d$) of a cubic lattice in d dimensions. The Langevin equations (in dimensionless form) are

$$\dot{x}_i=f(x_i)+g(x_i)\xi_i(t)-\frac{D}{2d}\sum_j (x_i-x_j)+\eta_i(t). \quad (1)$$

These stochastic differential equations are defined in the sense of Stratonovich calculus. The sum over j runs over the set of $2d$ nearest neighbors of site i. $\xi_i(t)$ and $\eta_i(t)$ are Gaussian white noises with zero mean values and the following correlation functions:

$$\begin{aligned}\langle \xi_i(t)\xi_j(t')\rangle&=2D_1\delta_{ij}\delta(t-t'),\\ \langle \eta_i(t)\eta_j(t')\rangle&=2D'\delta_{ij}\delta(t-t'),\\ \langle \xi_i(t)\eta_j(t')\rangle&=2\lambda\sqrt{D_1D'}\delta_{ij}\delta(t-t'),\end{aligned} \quad (2)$$

Since we use dimensionless variables x_i and t, the constants D, D_1, D', and λ are all dimensionless in Eqs. (1) and (2). Van den Broeck, Parrondo, and Toral [4] have reported the case without the additive noise $\eta_i(t)$. They mainly considered the nonequilibrium phase transition induced by multiplicative noise, and drew a conclusion that the transition possesses features similar to those observed at second order equilibrium phase transitions and is found to be reentrant. In addition, Van den Broeck *et al.* [5] also studied Eq. (1) for the case when the correlation intensity of $\xi_i(t)$ and $\eta_i(t)$ is zero [i.e., $\langle \xi_i(t)\eta_j(t')\rangle=0$], with $f(x_i)=x_i-x_i^3$ and $g(x_i)=x_i$. They pointed out that the phase transition would be shifted. We shall consider the effect of the correlation between the multiplicative noise $\xi_i(t)$ and the additive noise $\eta_i(t)$ on the phase transition. We can drop the subscript i in the following since Eq. (1) is similar for every site i. So it becomes

$$\dot{x}=f(x)+g(x)\xi(t)-D(x-\mu)+\eta(t), \quad (3)$$

where we have also introduced the Weiss mean-field approximation $\mu=\langle x\rangle=F(\mu)$, which have been extensively applied [2–5,9–13]. In the following, we give a method (this method is more simple than the one in Refs. [7,8]) by which we can derive the Fokker-Planck equation (FPE) for a stochastic equation when there is correlation between the two noises (Gaussian white noises).

We consider the stochastic differential equation

$$\dot{x}=h(x)+g_1(x)\xi(t)+g_2(x)\eta(t), \quad (4)$$

in which noises $\xi(t)$ and $\eta(t)$ are the same as those in Eq. (3). Let $\eta_0(t)=\eta(t)-\lambda\sqrt{D'/D_1}\xi(t)$. Obviously we get from Eq. (2) that $\langle \eta_0(t)\rangle=0$, $\langle \eta_0(t)\xi(t')\rangle=0$, and

$$\langle \eta_0(t)\eta_0(t')\rangle=2D'(1-\lambda^2)\delta(t-t'). \quad (5)$$

Then Eq. (4) can be transformed into the following form:

$$\dot{x}=h(x)+h_1(x)\xi(t)+g_2(x)\eta_0(t), \quad (6)$$

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where $h_1(x)=g_1(x)+\lambda\sqrt{D'/D_1}g_2(x)$; now the noises $\xi(t)$ and $\eta_0(t)$ are no longer correlated.

It is easy to obtain the FPE of Eq. (6) as follows:

$$\partial_t P(x,t)=-\partial_x h(x)P(x,t)+D_1\partial_x h_1(x)\partial_x h_1(x)P(x,t)+D'(1-\lambda^2)\partial_x g_2(x)\partial_x g_2(x)P(x,t). \quad (7)$$

Equation (7) accords with that obtained by the method of Refs. [7,8].

In order to obtain the FPE for Eq. (3), we compare Eq. (4) with Eq. (3) and get $h(x)=f(x)-D(x-\mu)$, $g_1(x)=g(x)$, and $g_2(x)=1$. Substituting these relations into Eq. (7), we obtain the FPE for Eq. (3):

$$\partial_t P(x,t)=-\partial_x[f(x)-D(x-\mu)]P(x,t)+D_1\partial_x f_1(x)\partial_x f_1(x)P(x,t)+D'(1-\lambda^2)\partial_x^2 P(x,t), \quad (8)$$

where $f_1(x)=g(x)+\lambda\sqrt{D'/D_1}$. Under the natural boundary condition, the stationary solution of Eq. (8) is [14,15]

$$P_{st}(x)=\frac{1}{N}\exp\left[\int^x dy\frac{f(y)-D_1g(y)g'(y)-D(y-\mu)-\lambda\sqrt{D_1D'}g'(y)}{D'+2\lambda\sqrt{D_1D'}g(y)+D_1g^2(y)}\right], \quad (9)$$

where N is a normalization constant. In terms of the Weiss mean-field approximation, we neglect the fluctuation in the neighboring sites. From Eq. (9) we can get

$$\mu=\langle x\rangle=F(\mu)=\int_{-\infty}^{\infty}xP_{st}(x)dx \quad (10)$$

(the order parameter $m=|\mu|$). In Ref. [4], Van den Broeck, Parrondo, and Toral predicted the existence of a symmetry-breaking phase transition with breaking of ergodicity by the Weiss mean-field approximation. As to the model studied in our paper, if $\lambda\neq 0$, as long as $f(x)$ is odd and $g(x)$ even it follows from Eq. (1) that any realization $\{x_i(t)\}$ is equally probable as $\{-x_i(t)\}$, so the symmetry-breaking phase transition exists; if $\lambda=0$, when $g(x)$ is odd or even and $f(x)$ odd, the symmetry-breaking phase transition can happen as in Ref. [4]. In order to illustrate the effect of the correlated noises on this phase transition, we give a possible example. Let $f(x)=-x(1+x^2)^2$ and $g(x)=1+x^2$. Substituting them into Eq. (9) and carrying out the integration, we can obtain the stationary probability density. We shall not present them here on account of their lengthiness.

We now turn to a more detailed analysis of Eq. (10). It is a self-consistent equation for μ,D,D_1, and D'. Obviously, the trivial solution $\mu=0$ always exists [for $\mu=0$, $P_{st}(x)$ is symmetrical]. With the appearance of multiple solutions, we can find "ordered" phases with an order parameter $m=|\langle x\rangle|\neq 0$ [the symmetry of Eq. (9) will be broken]. The critical condition should be

$$F'(\mu=0)=1 \quad \text{for } \lambda^2<1. \quad (11)$$

For the above special example, from Eq. (11) we can get

$$\int_{-\infty}^{\infty}dxG(x)\exp[-H(x)]=\int_{-\infty}^{\infty}dxG(x)\times\exp[-H(x)]\frac{Dx}{D_1}H_1(x) \quad (\text{for } \lambda<1), \quad (12)$$

where

$$G(x)=[D_1(1+x^2)^2+2\lambda\sqrt{D_1D'}(1+x^2)+D']^{C-1/2},$$

$$H(x)=\frac{1}{2D_1}(1+x^2)+\frac{E}{2A}\tan^{-1}\frac{D_1(1+x^2)+\lambda\sqrt{D_1D'}}{A},$$

$$H_1(x)=\frac{g}{2}\ln\frac{x^2+ax+d}{x^2-ax+d}+\frac{h}{2\sqrt{d-a^2/4}}\tan^{-1}\frac{x-a/2}{\sqrt{d-a^2/4}}+\frac{h}{2\sqrt{d-a^2/4}}\tan^{-1}\frac{x+a/2}{\sqrt{d-a^2/4}},$$

with $C=(\lambda/2D_1)\sqrt{D'/D_1}$, $E=D-(1-2\lambda^2)D'/D_1$, $A=[(1-\lambda^2)D_1D']^{1/2}$, $a=[2(2\lambda\sqrt{D'/D_1}+D'/D_1+1)^{1/2}-2(1+\lambda\sqrt{D'/D_1})]^{1/2}$, $d=(2\lambda\sqrt{D'/D_1}+D'/D_1+1)^{1/2}$, $h=1/2d$, and $g=h/a$. Here the condition D_1, $D'\neq 0$ for the validity of the Eq. (12) is necessary. In Fig. 1, we have plotted the phase transition line in the $D-D_1$ parameter plane in the light of Eq. (12). For simplicity we set $D'=1$, and $\lambda=0,0.5$, and 0.9, respectively. From this figure, we can find that the correlation between the additive noise and the multiplicative noise has influence on the phase transition. Firstly, the correlation always advances the transition to larger values of the spatially coupling constant D, i.e., the phase transition lines move to higher spatially coupling constant D. Secondly, along with the increase of λ the scope of D_1 for the transition when $0\leqslant D\leqslant 20$ becomes more and more narrow. Thirdly, when λ increases the transition line always moves toward the left. When D_1 is small this effect is feeble, while when D_1 is large it is strong. In addition, by analyzing Fig. 1 we find that the symmetry-breaking phase transition predicted by the Weiss mean-field approximation in our case has some similar properties to the one in Ref. [4]: the ordered phase only appears for a sufficiently strong spatially coupling constant D; the phase transition is reentrant.

Then let us compare the transition when $\lambda=0$ with the one in Ref. [4]. Because of the effect of the additive noises the phase transition in our case will happen at a smaller value of the multiplicative noise intensity and a larger value of the spatially coupling constant than the one in Ref. [4]. The

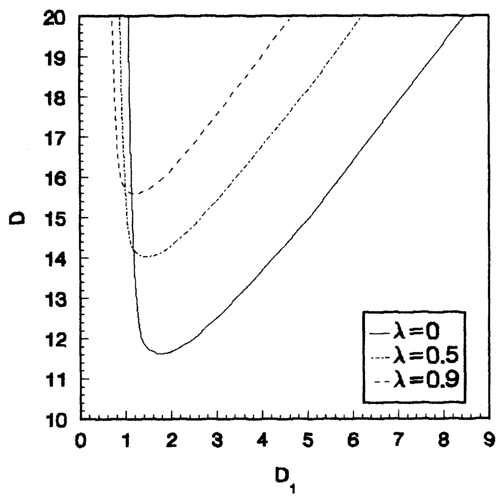

FIG. 1. The phase transition lines predicted by the mean-field theory in the D_1 vs D plane for several values of λ. $D'=1$ is fixed and $\lambda=0,0.5$, and 0.9, respectively.

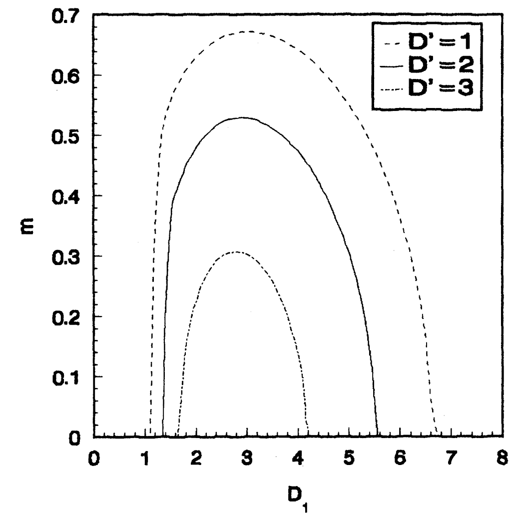

FIG. 3. m vs D_1 for $D=17$, according to the mean-field theory. $\lambda=0$ is fixed and $D'=1$, 2, and 3.

scope of the multiplicative noise intensity when $0\leqslant D\leqslant 20$ becomes more narrow (cf. Fig. 1 in Ref. [4] and Fig. 2 in our paper). By further study and analysis we find that when the intensity of the additive noise increases the scope of D_1 for the transition when $0\leqslant D\leqslant 20$ is more and more narrow. When $D'\to\infty$, the phase transition will disappear, which is like the one in the case of additive noise (in this case, the model does not undergo a phase transition). In Fig. 2 we have plotted the phase transition lines when $\lambda=0$ and $D'=1$, 2, and 3, respectively. Their positive nonzero solution $m=|\langle x\rangle|$ of Eq. (10) for $D=17$ is represented in Fig. 3. From Fig. 3 one finds that when the additive noise exists the mean-field region and the region of the values for D_1 in which the ordered phase is predicted are smaller (cf. the dashed line in Fig. 2 of Ref. [4]); with increasing D' the value of m becomes smaller and smaller, in the limit $D'\to\infty$ it tends to zero. Thus we can conclude that with the increase of D' the transition will become more and more indistinct, when $D'\to\infty$ the transition will disappear. Moreover in the limit $D'\to 0$, the phase transition line of $\lambda=0$ tends to the one in Ref. [4]. For the model (1), when one does not consider the additive noise (i.e., without the fluctuation of the internal conditions) it becomes the Langevin equation (1) of Ref. [4]. Although the formula (12) and the formula of μ obtained by us are gotten under the condition $D'\neq 0$, in the limit $D'\to 0$ they can tend to the corresponding ones in Ref. [4].

The positive nonzero solution $m=|\mu|$ is represented in Fig. 4. The figure shows that the transition is the second order one (which is similar to the one in Ref. [4]), since the order parameter m increases continuously. Thus the transi-

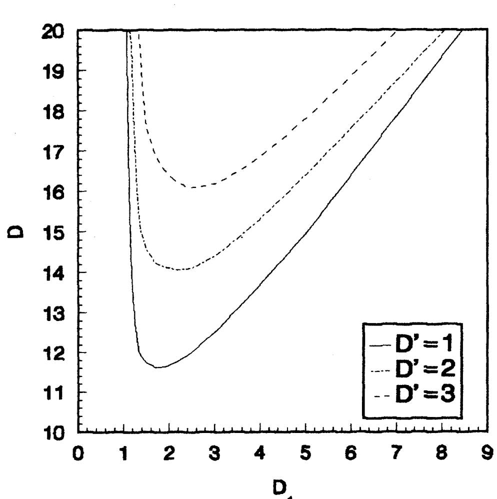

FIG. 2. The phase transition lines gotten by the mean-field theory in the case of $\lambda=0$ for several values of D'. $D'=1$, 2, and 3, respectively.

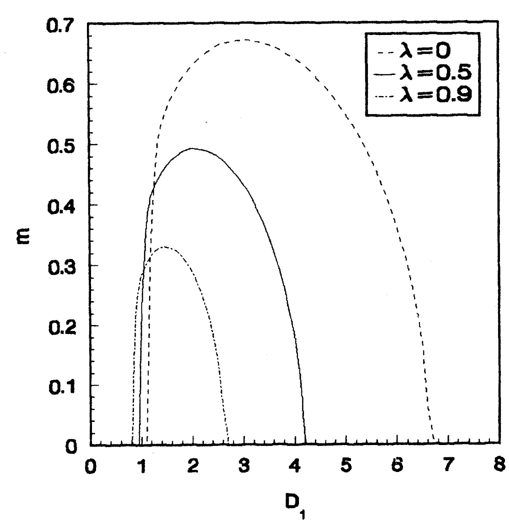

FIG. 4. m vs intensity of the multiplicative noise D_1 for $D=17$, according to the mean-field theory. $\lambda=0$, 0.5, and 0.9.

tion possesses the characteristic features: divergence of the correlation length and susceptibility, critical slowing down, and scaling behavior, etc. Moreover with the increase of λ the region for the order parameter m becomes smaller.

By analyzing the transitions for our model we find that so long as the model has the multiplicative noise, it will undergo a phase transition, which is reentrant. In the case of additive noise the transition does not exist. So we say that the multiplicative noise is the main role for the phase transition of the mean-field model (1).

In Ref. [5], Van den Broeck *et al.* have studied the case for $f(x_i)=\alpha x_i - x_i^3$ and $g(x_i)=x_i$ when the additive and multiplicative noises exist simultaneously in the case of $\lambda=0$. For the lattice model in our case, $g(x_i)$ must be even. If $g(x_i)$ is odd, $\mu=0$ is not the solution of Eq. (10), and the phase transition can not happen. Thus as for the model in Ref. [5], when the correlations between the additive noises and the multiplicative noises exist there would not be the nonequilibrium phase transition.

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